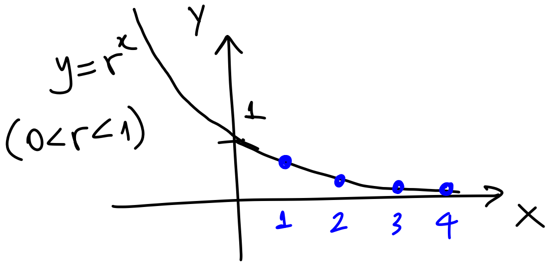
<u>Exercício:</u> Para quais valores de r a seq. $\{r^n\}$ é convergente?

- $\{1^n\} = \{1, 1, 1, 1, \dots\}$ (converge)

- $\{0^n\} = \{0, 0, 0, 0, \dots\}$ (converge)

- $0 < r < 1$: $\{r^n\}$, $f(x) = r^x$, $f(n) = r^n$

  ↑ exponencial

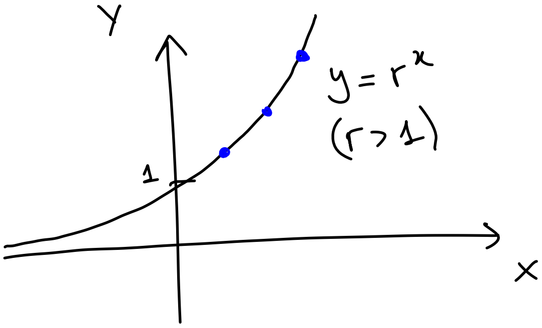


$$\lim_{x\to\infty} r^x = 0 \Rightarrow \lim_{n\to\infty} r^n = 0$$

(convergente)

- $r > 1$: $\{r^n\}$, $f(x) = r^x$, $f(n) = r^n$

  ↑ exp.

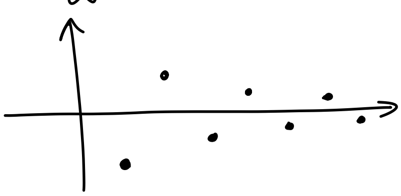


(diverge)

- $\{(-1)^n\} = \{-1, 1, -1, 1, -1, 1, \dots\}$ (diverge)

- $-1 < r < 0$: $\{r^n\}$, $|r| < 1$

$$\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0 \Rightarrow \lim_{n\to\infty} r^n = 0 \quad (r^n = (-1)^n |r|^n)$$

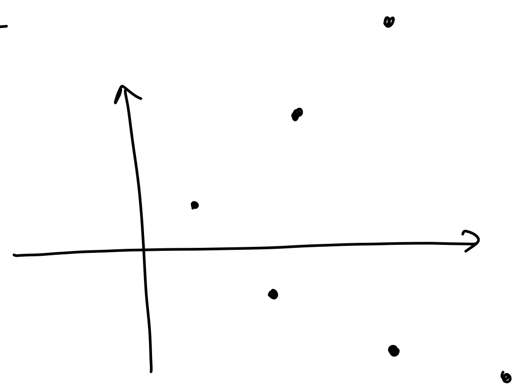


(converge)

- $r < -1$: $\{r^n\}$, $|r| > 1$

$$r^n = (-1)^n \cdot |r|^n$$

(diverge)



Portanto, $\lim_{n\to\infty} r^n = \begin{cases} 1, & r = 1 \\ 0, & -1 < r < 1 \\ \text{diverge}, & r \leq -1 \text{ ou } r > 1 \end{cases}$

converge em $(-1, 1]$.

Série

Dado uma seq. $\{x_n\}$, definimos a série

$$x_1 + x_2 + x_3 + \dots + x_n + \dots = \sum_{n=1}^{\infty} x_n .$$

Exemplo: 1) $1 + 2 + 3 + 4 + \dots + n + \dots = \sum_{n=1}^{\infty} n$

$S_1 = 1$

$\{S_n\}$

$S_2 = 1 + 2$

a soma S da série é tal que

$S_3 = 1 + 2 + 3$

$S = \lim_{n\to\infty} S_n$, se o lim. existir.

$\vdots$

$S_n = 1 + 2 + 3 + \dots + n$

$\uparrow$ seq. das somas parciais

$$S_n = \quad 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$$

$$+ \; S_n = n + (n-1) + (n-2) + (n-3) + \cdots + 2 + 1$$

$$2S_n = \underbrace{n + n + n + n + \cdots + n + n + n}_{n+1 \text{ vezes}} = n(n+1)$$

$$\Rightarrow S_n = \frac{n(n+1)}{2}$$

Gaus:

$1, 2, 3, \ldots, 50, 51, \ldots, 98, 99, 100$

$50 + 51 = 101$, $3 + 98 = 101$, $2 + 99 = 101$, $1 + 100 = 101$

$101 \cdot 50$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$$

série divergente

2) $$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

série convergente

$$\frac{1}{2^n} > 0$$

$$S_1 = \frac{1}{2} = 0{,}5$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{2+1}{4} = \frac{3}{4} = 0{,}75$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{4+2+1}{8} = \frac{7}{8} = 0{,}875$$

$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{8+4+2+1}{16} = \frac{15}{16} = 0{,}9375$$

$\vdots$

$$S_n = \frac{2^n - 1}{2^n} = \frac{2^n}{2^n} - \frac{1}{2^n} = 1 - \underbrace{\frac{1}{2^n}}_{>0} \xrightarrow{n\to\infty} 1$$

$$\left\{\frac{1}{2^n}\right\} \qquad \left\{1 - \frac{1}{2^n}\right\}$$

seq. das parcelas

seq. das somas parciais

3) $a \neq 0$:

$$a + ar + ar^2 + ar^3 + \dots + ar^n + \dots = \sum_{n=1}^{\infty} ar^{n-1}$$

- $r = 1$: $\quad S_n = a + a + a + \dots + a = n \cdot a \xrightarrow{n \to \infty} \infty$

- $r \neq 1$: $\quad S_n = a + \cancel{ar} + \cancel{ar^2} + \dots + \cancel{ar^{n-3}} + \cancel{ar^{n-2}} + \cancel{ar^{n-1}}$

$$-rS_n = \quad -\cancel{ar} - \cancel{ar^2} - \dots - \cancel{ar^{n-3}} - \cancel{ar^{n-2}} - \cancel{ar^{n-1}} - ar^n$$

$$S_n - rS_n = a - ar^n$$

$$\Rightarrow S_n(1-r) = a(1-r^n) \Rightarrow S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} \cdot (1-r^n)$$

$\therefore \{S_n\}$ é convergente se $-1 < r < 1$. Além disso,

$$\lim_{n \to \infty} S_n = \frac{a}{1-r}, \quad -1 < r < 1$$

Portanto, $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad -1 < r < 1$ (geométrica)

4) $$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \dots = \sum_{n=1}^{\infty} \frac{5 \cdot 2^{n-1}}{3^{n-1}} \cdot (-1)^{n-1} = \sum_{n=1}^{\infty} 5\left(-\frac{2}{3}\right)^{n-1}$$

é geométrica com $a = 5$ e $r = -\frac{2}{3}$. Logo, é convergente, pois $-1 < -\frac{2}{3} < 1$. Além disso, sua soma é

$$\frac{a}{1-r} = \frac{5}{1-\left(-\frac{2}{3}\right)} = \frac{5}{1+\frac{2}{3}} = \frac{5}{\frac{3+2}{3}} = 5 \cdot \frac{3}{5} = 3.$$

$$\boxed{\frac{x_n}{x_{n-1}} = r}$$

$$\frac{-\frac{40}{27}}{\frac{20}{9}} = -\frac{\cancel{40}}{27} \cdot \frac{9}{\cancel{20}} = -\frac{2}{3} \quad \Bigg| \quad \frac{\frac{20}{9}}{-\frac{10}{3}} = \frac{2\cancel{0}}{9}\left(-\frac{3}{1\cancel{0}}\right) = -\frac{2}{3} \quad \Bigg| \quad \frac{-\frac{10}{3}}{5} = \frac{-10}{3} \cdot \frac{1}{5} = -\frac{2}{3}$$